UFRJ-IF-FPC-010/96

Duality and an Operator Realization for the Fermi-Bose Transmutation in 3+1 Dimensions

E.C.Marino

Instituto de Física
Universidade Federal do Rio de Janeiro
Cx.P. 68528
Rio de Janeiro RJ 21945-970
Brazil

Abstract

We consider the Maxwell-Higgs system in the broken phase, described in terms of a Kalb-Ramond field interacting with the electromagnetic field through a topological coupling. We then study the creation operators of states which respectively carry a point charge and a closed magnetic string in the electromagnetic language or a point topological charge and a closed Kalb-Ramond charged string in the Kalb-Ramond dual language. Their commutation relation is evaluated, implying they satisfy a dual algebra and their composite possesses generalized statistics. In the local limit where the radius of the string vanishes, only Fermi or Bose statistics are allowed. This provides an explicit operator realization for statistical transmutation in 3+1D.

* Work supported in part by CNPq-Brazilian National Research Council.
E-Mail address: marino@if.ufrj.br

1) Introduction

The mechanism of statistical transmutation occurring in 2+1D for the composite state of a point charge and a magnetic vortex [1] has been generalized for 3+1D a few years ago. It was shown that the composite of a closed Nambu charged string and a point Kalb-Ramond vortex presented generalized statistics [2] in four-dimensional spacetime. Also the Chern-Simons mechanism of statistical transmutation [3] was shown to hold for stringlike objects in 3+1D [4]. More recently, in the framework of the Abelian Higgs Model (AHM), it was demonstrated that a closed magnetic (Nielsen-Olesen) string in the presence of a point charge also displayed the phenomenon of Fermi-Bose transmutation [5]. In this work we provide an operator realization for this fact. We consider the AHM in the broken phase, which can be described in terms of an antisymmetric tensor (Kalb-Ramond) gauge field interacting with the electromagnetic field by means of a topological coupling. In this language the topological charge of the Kalb-Ramond field becomes the electric charge and the Kalb-Ramond charge becomes the magnetic flux. A creation operator, σ, for the magnetic string states of the AHM was recently introduced in [6] and a creation operator, μ, for the topologically charged states of the Kalb-Ramond field was constructed some time before in [7]. Here, we obtain the operator dual to μ, in the Kalb-Ramond framework and show that it is nothing but the magnetic string operator σ, introduced in [6] rewritten in terms of the antisymmetric tensor gauge field. We evaluate their commutation relation and thereby explicitly show that the composite operator $\psi = \mu\sigma$ has generalized statistics. In the local limit where the radius of the closed string vanishes, we show that only Fermi or Bose statistics are allowed. In Kalb-Ramond language these composite states carry both a point topological charge (point Kalb-Ramond vortex) and a closed Kalb-Ramond charged string. In electromagnetic language, they carry both a point electric charge and a closed magnetic string. Through the composite operator ψ, therefore, we obtain an explicit realization for both the mechanisms discovered in [2] and [5].

2) The Maxwell-Higgs System and the Antisymmetric Tensor Gauge Field

Let us consider the Abelian Higgs Model, given by the lagrangian

$$\mathcal{L}_{AHM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu\phi|^2 + m^2|\phi|^2 - \frac{\lambda}{4}|\phi|^4 \tag{1}$$

Using the polar representation for the Higgs field $\phi = \frac{\rho}{\sqrt{2}}e^{i\theta}$ and integrating over θ in the approximation where the field ρ is made equal to a constant ρ_0 (large λ), we obtain the following lagrangian describing the effective dynamics of the electromagnetic field in the AHM [8, 6],

$$\mathcal{L}[A_\mu] = -\frac{1}{4}F_{\mu\nu}\left[1 + \frac{M^2}{-\Box}\right]F^{\mu\nu} \tag{2}$$

where $M = e\rho_0$. We can express any of the two terms above in terms of an antisymmetric tensor gauge field (Kalb-Ramond field), by means of the following identity in euclidean space

$$\int DB_{\mu\nu}\exp\left\{-\int d^4z\frac{1}{12}H_{\mu\nu\alpha}\left[\frac{A}{-\Box}\right]H^{\mu\nu\alpha} + \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}A_\mu\partial_\nu B_{\alpha\beta} + \mathcal{L}_{GF}\right\} =$$

$$\exp\left\{-\int d^4z\frac{1}{4}F_{\mu\nu}\left[\frac{1}{A}\right]F^{\mu\nu}\right\} \tag{3}$$

In the above expression, $H_{\mu\nu\alpha}$ is the field intensity tensor corresponding to the Kalb-Ramond field $B_{\mu\nu}$, A is an arbitrary linear operator and $\mathcal{L}_{GF}$ is the gauge fixing term. Making the choice $A = \frac{-\Box}{M^2}$, we can reproduce the second term of (2) and obtain the following equivalent lagrangian

$$\mathcal{L}[A_\mu, B_{\mu\nu}] = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{12}H_{\mu\nu\alpha}H^{\mu\nu\alpha} - \frac{1}{4}\epsilon^{\mu\nu\alpha\beta}B_{\mu\nu}F_{\alpha\beta} \tag{4}$$

where we have rescaled the antisymmetric field as $B_{\mu\nu} \to M\ B_{\mu\nu}$. This lagrangian describes the properties of the Abelian Higgs Model in the constant ρ approximation [5, 9]. The operator field equations corresponding to it are

$$\partial_\alpha H^{\alpha\mu\nu} = \frac{M}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{5}$$

and

$$\partial_\nu F^{\nu\mu} = \frac{M}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\nu B_{\alpha\beta} \tag{6}$$

From (6) we obtain

$$\partial_\alpha H^{\alpha\mu\nu} = \frac{\Box}{2M}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{7}$$

Consistency between (5) and (7) implies that the electromagnetic field satisfies the Proca equation which describes the well known screening of the broken phase of the theory. We see from (6) that the topological current of the antisymmetric field, namely, $J^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\nu B_{\alpha\beta}$ is proportional to the electric current: $j^\mu_{el} = e\rho_0 J^\mu$. Also, from (5), we conclude that the magnetic field is given by $B^i = \frac{1}{M}\partial_j \Pi^{ji}$, where $\Pi^{ji} = -H^{0ji}$ is the momentum canonically conjugated to the antisymmetric field B^{ij}.

3) The String and the Charge Creation Operators

Let us introduce now the operators which are going to create respectively magnetic string states and charged states. The charge creation operator, of course is the one which creates topological excitations, since the electric charge is identified with the topological charge. This has been introduced in [7]. In terms of the antisymmetric tensor field, it can written as

$$\mu(\vec{x},t) = \exp\left\{\frac{ib}{4\pi M}\int d^3\vec{\xi}\;\partial_i\varphi_j(\vec{\xi}-\vec{x})\Pi^{ij}(\vec{\xi},t)\right\} \tag{8}$$

or, in covariant form,

$$\mu(\vec{x},t) = \exp\left\{\frac{-ib}{4\pi M}\int d^3\xi_\mu\;\partial_\nu\varphi_\alpha(\vec{\xi}-\vec{x})H^{\mu\nu\alpha}(\vec{\xi},t)\right\} \tag{9}$$

In the above expressions $\varphi^\mu = (0,\vec{\varphi})$ and $\vec{\varphi} = \frac{1-\cos\theta}{r\sin\theta}\hat{\varphi}$, b is an arbitrary real parameter and a regulating procedure is implicit [7]. The equal-time commutator of μ with the topological carge (electric charge) $Q = \frac{M}{2}\epsilon^{ijk}\partial_i B_{jk}$ was evaluated in [7], yielding the result

$$[Q,\mu] = b\mu \tag{10}$$

This shows that the operator μ carries b units of electric charge.

Let us introduce now the (closed) magnetic string operator. This is given by

$$\sigma(C,t) = \exp\left\{\frac{iaM}{2}\int_{S(C)} d^2\xi_{\mu\nu}\frac{\partial_\alpha H^{\alpha\mu\nu}}{-\Box}\right\} \tag{11}$$

or

$$\sigma(C,t) = \exp\left\{\frac{-iaM}{2}\int_{S(C)} d^2\xi_{ij}B^{ij} + gauge\ terms\right\} \tag{12}$$

In the above expressions, a is an arbitrary real number and $S(C)$ is a space-like surface bounded by the closed string at C. $d^2\xi_{ij}$ is the surface element of $S(C)$, the directions i,j being along the surface. The gauge terms in (12) guarantee the gauge invariance of σ which is explicit in (11). Later on, it will become clear that both the correlation functions and commutation rules of σ are independent of the surface S: they just depend on C. The generalization for an open string is straightforward. The operator $\sigma(C)$ creates a magnetic string along the closed curve C. In order to prove this, let us consider the magnetic flux operator along a surface R. This is given by

$$\Phi_R = \int_R d^2\xi^i B^i = \int_R d^2\xi^i \frac{1}{M}\partial_j \Pi^{ji} \tag{13}$$

where we used the expression of the magnetic field B^i in terms of the antisymmetric tensor momentum. Writing $\sigma \equiv e^{\alpha}$, we obtain $[\Phi_R, \sigma] = \sigma[\Phi_R, \alpha]$ since the latter commutator is a c-number. Indeed, using (12), (13) and the canonical commutation rules of the antisymmetric field, we immediately get

$$[\Phi_R, \alpha] = a\int_R d^2\eta^j \oint_C d\xi^j \delta^3(\xi - \eta) \tag{14}$$

The above integrals give ± 1 whenever the curve C pierces the surface R in the positive or negative sense, respectively. Otherwise they vanish. Hence, we if we choose the magnetic flux surface and the string in such a way that it pierces the surface positively, we get

$$[\Phi_R, \sigma] = a\ \sigma \tag{15}$$

This shows that the σ operator carries a units of magnetic flux along the curve C. Actually, substituting (7) in (11) we obtain precisely the magnetic string operator introduced in [6].

4) Dual Algebra and Statistical Transmutation

Let us determine now the commutation rule between the operators $\sigma \equiv e^{\alpha}$ and $\mu \equiv e^{\beta}$. Since $[\beta, \alpha]$ is a c-number, we have

$$\mu(\vec{x},t)\sigma(C_y,t) \equiv \sigma(C_y,t)\mu(\vec{x},t)e^{[\beta,\alpha]} \tag{16}$$

Using (8), (12) and the canonical commutation rules of the antisymmetric tensor field, we get

$$[\beta(\vec{x},t), \alpha(C_y,t)] = i\frac{ab}{8\pi}\int_{S(C_y)} d^2\xi^i \epsilon^{ijk}(\partial_j\varphi_k - \partial_k\varphi_j)(\vec{\xi} - \vec{x}) \tag{17}$$

where $d^2\xi^i = \frac{1}{2}\epsilon^{ijk}d^2\xi^{jk}$ is the surface element of $S(C_y)$. The identity

$$(\partial_j\varphi_k - \partial_k\varphi_j)(\vec{\xi} - \vec{x}) \equiv \epsilon^{ijk}\partial_i\left[\frac{1}{|\vec{\xi} - \vec{x}|}\right] \tag{18}$$

allows us to obtain the result

$$[\beta(\vec{x},t), \alpha(C_y,t)] = i\frac{ab}{4\pi}\int_{S(C_y)} d^2\xi^i \frac{(x-\xi)^i}{|\vec{x} - \vec{\xi}|^3} = i\frac{ab}{4\pi}\Omega(\vec{x};C_y) \tag{19}$$

where $\Omega(\vec{x};C_y)$ is the solid angle comprised between $\vec{x}$ and the curve C_y. As a consequence we get

$$\mu(\vec{x},t)\sigma(C_y,t) = \exp\left\{i\frac{ab}{4\pi}\Omega(\vec{x};C_y)\right\}\sigma(C_y,t)\mu(\vec{x},t) \tag{20}$$

This is an algebra analogous the order-disorder algebra [10, 11, 12] and characterizes the charge and magnetic string operators μ and σ as dual to each other. Observe that (20) is surface independent. Also the correlation functions of σ, in the electromagnetic language, were shown to be surface independent [6].

We can now construct the creation operator for the composite state carrying charge and magnetic flux along a closed curve C. We can choose the curve C_x to be a circle of radius R centered at $\vec{x}$ and place the charge in the center, namely,

$$\psi(x;C_x;t) = \lim_{\vec{x}\to\vec{y}}\mu(\vec{x},t)\sigma(C_y,t) \tag{21}$$

Using the fact that $\Omega(\vec{x}; C_y) - \Omega(\vec{y}; C_x) = 4\pi\epsilon\left(\Omega(\vec{x}; C_y)\right)$, where $\epsilon(x)$ is the sign function, we obtain from (20)

$$\psi(x; C_x; t)\psi(y; C_y; t) = e^{i\ ab\epsilon(\Omega(\vec{x};C_y))}\psi(y; C_y; t)\psi(x; C_x; t) \tag{22}$$

This relation shows that the composite charge-magnetic string state possesses statistics $s = \frac{ab}{2\pi}$ thereby providing an explicit operator realization for the statistical transmutation of strings in the presence of point sources, discovered in [4, 5]. In the local limit when we make the radius of the circle to vanish, the exponential factor becomes a constant and the only consistent possibilities for ab are either $ab = \pi$ or $ab = 2\pi$ (and their corresponding multiples), meaning that ψ can only be a fermion or a boson in the local case. This agrees with the well known fact that only fermionic or bosonic local fields can exist in 3+1D.

5) Remarks on Bosonization

The operator construction presented here shows that it is possible to describe fermionic states in terms of bosonic gauge fields in 3+1D. This is in agreement with the bosonization of the action and of the current in four dimensional spacetime [13]. Of course, the crucial problem of bosonization is to find the correct operators which would describe the correlation functions of a specific fermionic thery in the framework of the corresponding bosonic one. Bosonized lagrangians corresponding to fermionic theories have been obtained [13]. The operators studied here may be a step towards the obtainment of a full bosonization in 3+1D. It is interesting to note that if this kind of construction would prove to be correct, say, for the description of electrons, then these would come up as closed string states bound to a charge. The relation between charge and magnetic flux responsible for the fermionic statistics would at the same time explain the quantization of charge.

Acknowledgements

I am grateful to R.Banerjee for calling my attention to reference [5].

References

[1] F.Wilczek, *Phys. Rev. Lett.* **49** (1982) 957

[2] R.Gambini and R.Setaro, *Phys. Rev. Lett.* **65** (1990) 2623

[3] A.M.Polyakov, *Mod. Phys. Lett.* **A3** (1988) 325

[4] X.Fustero, R.Gambini and A.Trias *Phys. Rev. Lett.* **62** (1989) 1964

[5] H.Fort and R.Gambini *Phys. Lett.* **B63** (1996) 226

[6] E.C.Marino, *Phys. Rev.* **D53** (1996) 1001

[7] E.C.Marino, *Int. J. Mod. Phys.* **A9** (1994) 4009

[8] E.C.Marino, *Int. J. Mod. Phys.* **A10** (1995) 4311

[9] A.P.Balachandran and P.Teotonio-Sobrinho, *Int. J. Mod. Phys.* **A9** (1994) 1569; S.Coleman, J.Preskill and F.Wilczek, *Nucl. Phys.* **B378** (1992) 175

[10] L.P.Kadanoff and H.Ceva, *Phys. Rev.* **B3** (1971) 3918

[11] G. 't Hooft, *Nucl. Phys.* **B138** (1978) 1

[12] E.C.Marino and J.A.Swieca, *Nucl. Phys.* **B170 [FS1]** (1980) 175; E.C.Marino, B.Schroer and J.A.Swieca, *Nucl. Phys.* **B200 [FS4]** (1982) 499; E.C.Marino, *Phys. Rev.* **D38** (1988) 3194; E.C.Marino and J.E.Stephany Ruiz, *Phys. Rev.* **D39** (1989) 3690

[13] C.Burgees, C.Lütken and F.Quevedo, *Phys. Lett.* **B336** (1994) 18; R.Banerjee, *Nucl. Phys.* **B465** (1996) 157; P.A.Marchetti, *"Bosonization and Duality in Condensed Matter Systems"*, hep-th/9511100 (1995); K.Ikegami, K.Kondo and A.Nakamura, *Progr. Th. Phys.* **95** (1996) 203